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Random Surfaces from Hierarchical Deposition of Debris with Alternating Rescaling Factors[1]

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An analytical study of surface profiles that result from hierarchical random impact of debris on the line is performed in terms of logarithmic fractals theory. The hierarchical random deposition model is extended for the case of time-dependent probabilities P (for positioning a hill on the surface) and Q (for digging a hole) and spatial rescaling factor λ. The periodic deposition model is solved exactly, and the logarithmic fractal roughness of the surface profile is found to be robust with respect to time-dependent perturbations. The fractal amplitudes associated with the proliferation of the surface length are compared with those calculated in the static regime and are shown to have a nontrivial interaction. It is verified that amplitude repulsion, attraction, neutrality, and auto-repulsion take place. The transient regime is also studied and is shown to have exponential decay towards the asymptotic regime. Special attention is devoted to the case of alternating rescaling factors, for which new results are derived.

KEY WORDS: fractals; hierarchical models; random deposition models.

1. GENERAL INTRODUCTION

A standard Euclidean geometry restricts human imagination, reducing all the images we can see or feel to combinations of circles, cones, spheres, cubes, etc. But most natural objects are so irregular and fragmented as to deserve being called geometrically chaotic. Nature exhibits not simply a higher degree, but a different level of complexity, compared to Euclidean geometry.

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In 1975, Mandelbrot introduced the concept of fractal geometry to characterize the objects of nature quantitatively and to help with an appreciation and understanding of their *underlying* regularity [1]. Fractals are more than fancy, computer generated patterns. The coastline of an island, a river network, or the structure of a cabbage can be best described as fractals. Yet, there is no generally accepted definition of a fractal (below will be given some rigorous "mathematical" definition). It can be defined loosely as a shape made of parts similar to the whole in some sense, i.e., fractals tend to be scaling, implying that the degree of their irregularity and/or fragmentation is identical at all scales.

The simplest way to construct a fractal is to repeat a given operation over and over again deterministically. The classical example of such a fractal is the Cantor set. It is created by dividing a line into n equal pieces and removing $(n-m)$ of the parts created and repeating the process with m remaining pieces ad infinitum [1, 2].

However, fractals that occur in nature occur through a continuous *random* process. For example, we can imagine selecting a line randomly at a given rate, and dividing it randomly. Starting with a line of given length, we obtain an infinite number of points, whose properties appear to be statistically self-similar and characterized by a fractal dimension. This principle can be extended to two dimensions to understand fractals in nature that have both size and shape. But it seems that the phenomenon cannot be described by a single fractal dimension—infinitely many are required [3]. Thus, the ancient notions of dimension and of symmetry play a central role in fractal theory.

The fact that basic fractals are dimensionally discordant can serve to transform the concept of fractal from an intuitive to a mathematical one. According to a definition, given by Mandelbrot, a fractal is a set for which the Hausdorff–Besikovitch dimension D_F (Mandelbrot calls D_F a fractal dimension) strictly exceeds the topological dimension D_T. For example, the original Cantor set is a fractal, because $D_F = \ln 2/\ln 3 > 0$, while $D_T = 0$.

Yet borderline cases with $D_F = D_T$ intuitively deserve to be called fractals as well. For example, one is very reluctant to call the so-called Devil's staircase a nonfractal, since it is broken on many length scales in an obvious fashion. A subset of fractals with $D_F = D_T$ is referred to as *logarithmic* fractals [4], since the Hausdorff measure for such a fractal is logarithmic:

$$h(\rho) = \rho^{D_F}(\ln(1/\rho))^{\Delta_1} \tag{1.1}$$

where ρ is the ruler length, $D_F = D_T$, and we assume subdimension $\Delta_1 = -1$. Further subdimensions, associated with multiple-logarithmic behavior, can also occur, but are not taken into account here.

These marginal fractals have attracted unfairly sparse attention from physicists, while these models can, for example, describe the patterns that arise in random sequential deposition of a mixture of particles with a continuous distribution of sizes on a finite substrate. One can readily see that with a continuous distribution of sizes the system does not reach a jamming limit, but instead creates a scale invariant pattern that can be described as a fractal [5], since the system gains its ergodic nature. Models of random deposition can rarely be solved analytically. In this paper we study an exactly solvable model of hierarchical random deposition of debris on a surface.

2. PERIODIC RANDOM DEPOSITION MODEL

A model for random hierarchical deposition of debris has recently been proposed [4]. It features fragments of different sizes impacting on a surface and transforming it into a fractal landscape with unusual geometrical properties. Hierarchy means that the larger objects hit the surface first. This kind of hierarchy arises naturally, for example, in air or another viscous medium where due to friction the velocities vary according to fragment weight, size, and shape.

A rescaling factor λ controls the size reduction from one generation to the next. In case of deposition on a line, for example, the number of fragments N of linear size s, which follows a simple hyperbolic law, similar to the distribution found for meteors hitting celestial bodies in our solar system, is given by:

$$N(s) = \lambda^2 N(s/\lambda) \tag{2.2}$$

The deposition process is considered to be random, and the deposition segments are independent.

The falling fragments are squares of different sizes, representing the debris of some "explosion." The deposition of a square hill is described by probability P. In addition to different sizes and shapes, fragments can possess different hardness or reactivity. That means that falling segments can also "dig" a square hole with probability Q (with $P+Q \leqslant 1$). Thus, the substrate can be characterized by inhomogeneous penetrability.

Originally [4] the probabilities P and Q, and rescaling factor λ were assumed to be the fixed parameters. In the present paper we consider the dynamic generalization [6] of the "static" model [4] for the case of time-dependent parameters: P, Q, and λ. It is clear that this "periodic" model is more realistic from a physical point of view. Assuming P, Q, and rescaling factor λ time-dependent, we can model situations with deposition of debris

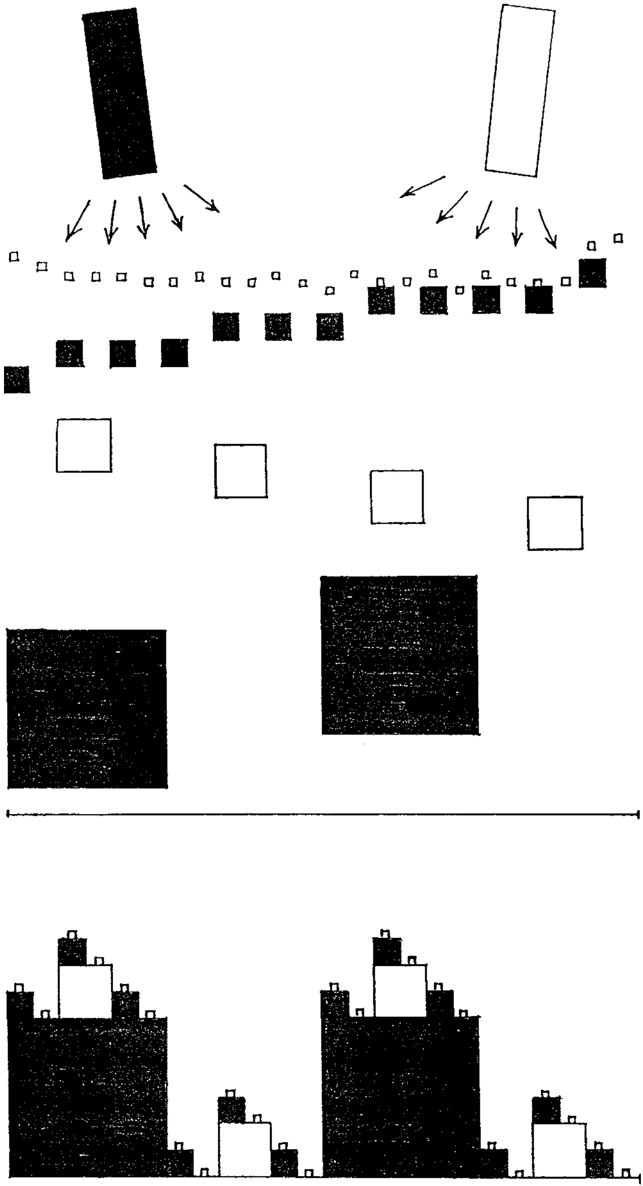

Fig. 1a, b. The deterministic case of the deposition of debris, injected from two sources. The rescaling factor λ alternates between two values: $\lambda_1 = 2$ and $\lambda_2 = 3$. Figure 1b exemplifies the resulting surface profile after four generations (2 periods). The profile has been repeated once in the horizontal direction, along the substrate.

originating from multiple sources. Each source produces its own "particles" with specific characteristics: mass, size distribution, and impact capacity. This means that each source can be characterized by its own P, Q, and rescaling factor λ. Thus, the process with period 2 can correspond to two sources, which are active serially and independently. For example, we can think of two alternating beams of laser-evaporated atomic clusters that hit a target surface. This can nowadays be achieved experimentally [7, 8]. Another attractive characteristic of the "periodic" model is that it also possesses logarithmic fractal properties and can still be solved analytically.

The deterministic case for the rescaling factor, alternating between $\lambda_1 = 2$ and $\lambda_2 = 3$, is shown in Fig. 1. Two sources, injecting particles of two types, are depicted on Fig. 1a. Figure 1b illustrates the surface profile after four generations in the above mentioned periodic process and a random process with probability of deposition of square hills alternating between

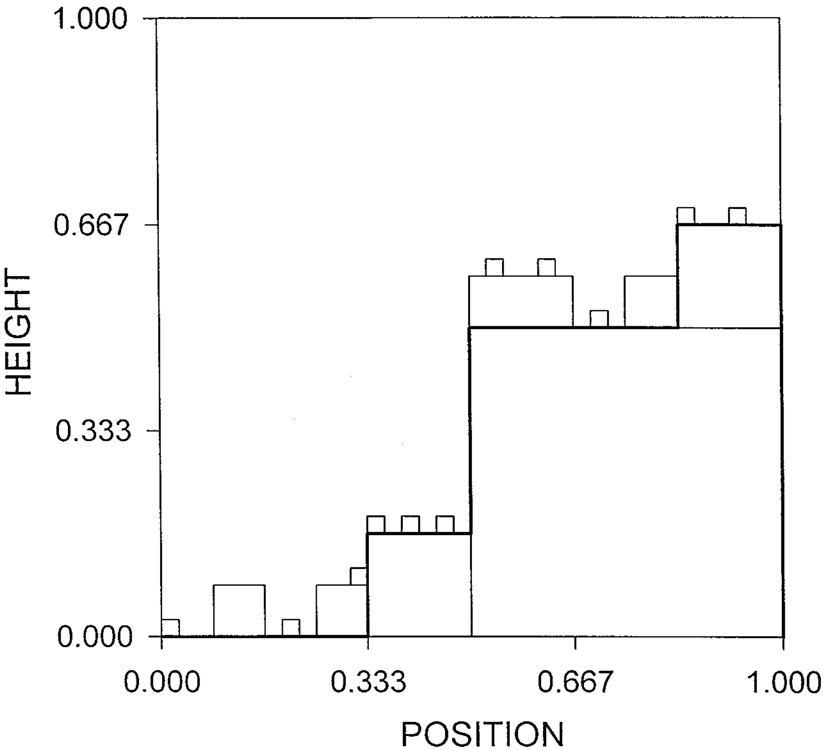

Fig. 2. A random landscape corresponding to $\lambda_1 = 2$ and $\lambda_2 = 3$, and $P_1 = 1/2$ and $P_2 = 1/3$. The result after four generations (2 periods) is shown. Also shown are the profiles of the first, second (thick line), and third generations.

$P_1 = 1/2$ and $P_2 = 1/3$ is shown in Fig. 2 for the same choice of rescalings $\lambda_1 = 2$ and $\lambda_2 = 3$.

3. ALTERNATING *P* AND *Q*: ASYMPTOTIC REGIME

Let us assume that P and Q alternate between two sets of values (P_1, Q_1) and (P_2, Q_2), while the rescaling factor is fixed: $\lambda_1 = \lambda_2 = \lambda$, i.e., we consider the periodic process with a period equal to two generations. We take $\lambda \geqslant 3$. The case $\lambda = 2$ is special [4].

In the case of fixed P and Q, the analytic solution for the average length of the profile (length increment) in the nth generation is derived along the lines of previous work [4] and for the periodic boundary conditions has the following form:

$$\Delta L_n = \lambda^{-n} \sum_{i=1}^{\lambda^n} 2(P(1-P) + Q(1-Q))\{1 - W_i^{(n-1)}\} \tag{3.3}$$

The sum of $W_i^{(n-1)}$ over i equals the average number of points occupied by at least one wall (vertical segment placed on points between horizontal segments) after $n-1$ generations:

$$\sum_{i=1}^{\lambda^n} W_i^{(n-1)} = \sum_{m=1}^{n-1} \lambda^m \frac{\Delta L_m}{(\alpha' + 2\alpha'')} \tag{3.4}$$

In this expression the functions α' and α'' are defined as follows,

$$\begin{aligned} \alpha' &= (P+Q)(1-(P+Q))/(P(1-P) + Q(1-Q) - PQ) \\ \alpha'' &= PQ/(P(1-P) + Q(1-Q) - PQ) \end{aligned} \tag{3.5}$$

While depositing progresses from one generation n to the next $(n+1)$, the average length increment converges rather fast to some constant asymptotic value ΔL_∞, which does not depend on n:

$$\Delta L_\infty(P, Q, \lambda) = \frac{2(P(1-P) + Q(1-Q))}{1 + 2(P(1-P) + Q(1-Q) - PQ)/(\lambda - 1)} \tag{3.6}$$

This is a very important property of so-called logarithmic fractals.

In the case of dynamical perturbations, we should rewrite this formula taking into account that P and Q alternate between two sets of values. Thus, in the periodic deposition model [6], instead of one asymptotic value two n-independent length increments arise: $\Delta L_{\infty,\,\mathrm{odd}}$ and $\Delta L_{\infty,\,\mathrm{even}}$:

$$\begin{aligned} \Delta L_{\infty,\,\mathrm{odd}} &= \Delta L_{\infty,\,1}\left\{1+\frac{\lambda(\beta_1-\beta_2)}{\lambda^2-1+\beta_1+\beta_2-\beta_1\beta_2}\right\} \\ \Delta L_{\infty,\,\mathrm{even}} &= \Delta L_{\infty,\,2}\left\{1-\frac{\lambda(\beta_1-\beta_2)}{\lambda^2-1+\beta_1+\beta_2-\beta_1\beta_2}\right\} \end{aligned} \tag{3.7}$$

where

$$\beta_i = 2(P_i(1-P_i)+Q_i(1-Q_i)-P_iQ_i) \tag{3.8}$$

and

$$\begin{aligned} \Delta L_{\infty,\,1} &= \Delta L_{\infty}(P_1, Q_1, \lambda) \\ \Delta L_{\infty,\,2} &= \Delta L_{\infty}(P_2, Q_2, \lambda) \end{aligned} \tag{3.9}$$

Thus, in the time-dependent case we have the coexistence of two converging processes: one with odd n and one with even n. The periodic process again leads to a logarithmic fractal with effective rescaling factor $\lambda_1\lambda_2$ (which equals λ^2) and a fractal amplitude equal to the sum $\Delta L_{\infty,\,\mathrm{odd}} + \Delta L_{\infty,\,\mathrm{even}}$ on the time scale of one period. This shows that the basic property of the model is robust with respect to the dynamical perturbation.

Since the values for static and dynamical asymptotic amplitudes do not coincide (for $\beta_1 \neq \beta_2$), one can describe the periodic process in terms of "interaction." For $\beta_1 \neq \beta_2$ either repulsion or attraction of amplitudes takes place. Note that in view of the "$+$" and "$-$" signs in front of the amplitude shifts implied by Eq. (3.7), the amplitudes are shifted from there bare values in opposite directions. The condition $\beta_1 = \beta_2$ corresponds to so-called neutrality, when amplitudes $\Delta L_{\infty,\,\mathrm{odd}}$ and $\Delta L_{\infty,\,\mathrm{even}}$ do not "interact," i.e., do not move away from their bare values for the static case.

The nature of amplitude interaction can be exemplified by investigating in more detail the special period-2 case (Fig. 3). We fix $P_1 = 0.15$, $Q_1 = 0.7$, $Q_2 = 0.2$, and $\lambda = 3$ and plot P_2-dependences for the fractal amplitudes of static and periodic processes. The bare amplitudes are given by the thin lines, and the thick lines show the result of the interaction. Thus, $\Delta L_{\infty,\,\mathrm{odd}}$ and $\Delta L_{\infty,\,\mathrm{even}}$ coincide for $P_2 \cong 0.213$ and $P_2 \cong 0.755$. The static amplitudes intersect only for $P_2 \cong 0.269$. For the neutral points we have $P_2 \cong 0.104$ and $P_2 \cong 0.696$.

4. ALTERNATING *P* AND *Q*: TRANSIENT REGIME

For the periodic model described in Section 3, it is also interesting to study the transient regime, which again can be calculated exactly. The transient regime refers to a law, according to which the average length

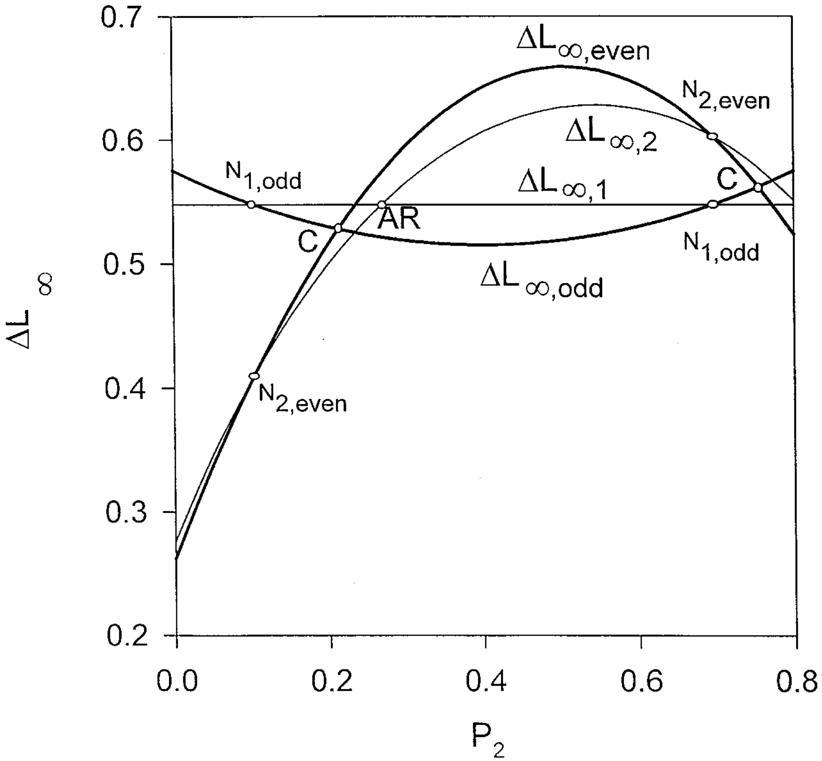

Fig. 3. Amplitude repulsion, neutrality, attraction, coincidence, and auto-repulsion. In this period-2 process with $\lambda = 3$ we have set $Q_1 = 0.7$, $Q_2 = 0.2$, and let P alternate between $P_1 = 0.15$ and P_2. For $P_2 < 0.104$ there is amplitude repulsion. Then, at $P_2 \cong 0.104$ the condition of neutrality is satisfied, and the amplitudes do not move away from their bare static values (points N). For larger P_2 a regime of amplitude attraction is found, culminating at point C in the coincidence of the amplitudes. At somewhat larger P_2 the amplitudes again repel. At the special point of auto-repulsion AR the bare static amplitudes coincide, but the amplitudes repel each other fairly strongly in the period-2 process. The same scenario is repeated in the vicinity of the second point of coincidence at $P_2 \cong 0.755$.

increment of the profile, ΔL_n, converges to its asymptotic value ΔL_∞. In the static case this law is exponential.

We can define the transients as:

$$\begin{aligned} \epsilon_n &\equiv \Delta L_n - \Delta L_{\infty,\,\mathrm{odd}} \\ \epsilon_{n+1} &\equiv \Delta L_{n+1} - \Delta L_{\infty,\,\mathrm{even}} \end{aligned} \tag{4.10}$$

where we assume odd values of n.

After some algebra the result can be written in the following form [6]:

$$\Delta L_n = \Delta L_{\infty,\,\mathrm{odd}} \left\{ 1 + \frac{\beta_1 + \lambda\beta_2 - \beta_1\beta_2}{(\lambda-1)(\lambda+1-\beta_2)} \left(\frac{\lambda^2}{(1-\beta_1)(1-\beta_2)} \right)^{(1-n)/2} \right\}$$

$$\Delta L_{n+1} = \Delta L_{\infty,\,\mathrm{even}} \left\{ 1 + \frac{\beta_1 + \lambda\beta_2 - \beta_1\beta_2}{(\lambda-1)(\lambda+1-\beta_1)} \left(\frac{\lambda^2}{(1-\beta_1)(1-\beta_2)} \right)^{(1-n)/2} \left(\frac{1-\beta_1}{\lambda} \right) \right\} \tag{4.11}$$

From here one can see that the transient decays exponentially fast but not necessarily monotonically. The odd and even transients interact in the sense that they differ from the transients of the respective static processes. This is quite natural, however, since the asymptotic amplitudes interact. It can be shown that in the absence of interaction (neutrality condition), the period-2 transient alternates exactly between the two transients for the respective static processes.

5. ALTERNATING RESCALING: ASYMPTOTIC REGIME

As a final example we consider in detail a periodic process with the rescaling factor λ alternating between two values: λ_1 and λ_2. In a previous paper this case was treated for the special choice $\lambda_2 = \lambda_1^2$ and $Q = 0$ only [6]. For simplicity we will be keeping the probabilities P and Q fixed: $P_1 = P_2 = P$ and $Q_1 = Q_2 = Q$. The logarithmic fractal feature is robust with respect to alternating rescaling as well. For $P = 0$ or $Q = 0$ our calculation is valid for $\lambda_1 \geqslant 2$ and $\lambda_2 \geqslant 2$, while for $P \neq 0 \neq Q$ the range of validity is reduced to $\lambda_1 \geqslant 3$ and $\lambda_2 \geqslant 3$, due to wall-leveling [4].

We start out from the coupled implicit equations, originating from Eq. (3.3), assuming odd n:

$$\Delta L_n = \frac{1}{N(n)} \sum_{i=1}^{N(n)} 2(P(1-P) + Q(1-Q))\{1 - W_i^{(n-1)}\}$$

$$\Delta L_{n+1} = \frac{1}{N'(n)} \sum_{i=1}^{N'(n)} 2(P(1-P) + Q(1-Q))\{1 - W_i^{(n)}\} \tag{5.12}$$

where

$$\sum_{i=1}^{N(n)} W_i^{(n-1)} = \sum_{m=1,\,\mathrm{odd}}^{n-2} N(m) \frac{\Delta L_m}{\alpha' + 2\alpha''} + \sum_{m=2,\,\mathrm{even}}^{n-1} N'(m-1) \frac{\Delta L_m}{\alpha' + 2\alpha''}$$

$$\sum_{i=1}^{N'(n)} W_i^{(n)} = \sum_{m=1,\,\mathrm{odd}}^{n} N'(m) \frac{\Delta L_m}{\alpha' + 2\alpha''} + \sum_{m=2,\,\mathrm{even}}^{n-1} N'(m-1) \frac{\Delta L_m}{\alpha' + 2\alpha''} \tag{5.13}$$

where $N(n) = \lambda_1^{(n+1)/2} \lambda_2^{(n-1)/2}$ and $N'(n) = \lambda_1^{(n+1)/2} \lambda_2^{(n+1)/2}$.

In order to obtain the asymptotic behavior, we replace the nonzero length increments by their fixed point values: $\Delta L_n \to \Delta L_{\infty,\,\mathrm{odd}}$ and $\Delta L_{n+1} \to \Delta L_{\infty,\,\mathrm{even}}$. After some algebraic calculations we obtain the following coupled equations:

$$\begin{aligned} \Delta L_{\infty,\,\mathrm{odd}} &= a - \frac{\beta}{\lambda_1\lambda_2 - 1}\,\Delta L_{\infty,\,\mathrm{odd}} - \frac{\beta\lambda_2}{\lambda_1\lambda_2 - 1}\,\Delta L_{\infty,\,\mathrm{even}} \\ \Delta L_{\infty,\,\mathrm{even}} &= a - \frac{\beta\lambda_1}{\lambda_1\lambda_2 - 1}\,\Delta L_{\infty,\,\mathrm{odd}} - \frac{\beta}{\lambda_1\lambda_2 - 1}\,\Delta L_{\infty,\,\mathrm{even}} \end{aligned} \tag{5.14}$$

where we set $a = 2(P(1-P) + Q(1-Q))$ and β is defined in accordance with Eq. (3.8).

Finally we obtain for the asymptotic amplitudes in the periodic model:

$$\begin{aligned} \Delta L_{\infty,\,\mathrm{odd}} &= \Delta L_{\infty,\,1}\left(1 + \frac{(\lambda_2 - \lambda_1)\,\beta(1-\beta)}{(\lambda_1 - 1)(\lambda_1\lambda_2 - 1 + 2\beta - \beta^2)}\right) \\ \Delta L_{\infty,\,\mathrm{even}} &= \Delta L_{\infty,\,2}\left(1 - \frac{(\lambda_2 - \lambda_1)\,\beta(1-\beta)}{(\lambda_2 - 1)(\lambda_1\lambda_2 - 1 + 2\beta - \beta^2)}\right) \end{aligned} \tag{5.15}$$

where $\Delta L_{\infty,\,1}$, $\Delta L_{\infty,\,2}$ are correspondingly $\Delta L_\infty(P, Q, \lambda_1)$ and $\Delta L_\infty(P, Q, \lambda_2)$ from Eq. (3.6).

The amplitudes for an alternating-λ process with $\lambda_1 = 2$ and $\lambda_2 = 4$ are shown in Fig. 4. The P-dependence of the static and dynamic fractal amplitudes for $Q = 0$ is depicted. Amplitude attraction takes place for all P. We considered $Q = 0$ because only under this condition λ can be equal to 2, without involving wall-leveling corrections [4].

Figure 5 exemplifies a similar process with $\lambda_1 = 3$ and $\lambda_2 = 8$, and Q nonzero: $Q = 0.3$. Hence, the physical range for probability P is $0 \leqslant P \leqslant 0.7$. Again only attraction between the amplitudes occurs. For alternating re-scalings there is always amplitude attraction, as is seen from the structure of Eq. (5.15). For instance, for $\lambda_2 > \lambda_1$, we have $\Delta L_{\infty,\,\mathrm{odd}} > \Delta L_{\infty,\,1}$ and $\Delta L_{\infty,\,\mathrm{even}} < \Delta L_{\infty,\,2}$. Furthermore, from Eq. (3.6) follows $\Delta L_{\infty,\,1} < \Delta L_{\infty,\,2}$, so that the dynamic amplitudes lie between the static ones.

6. CONCLUSIONS

In this paper we have analyzed a model for hierarchical random deposition of debris, taking into account various possible dynamical extensions.

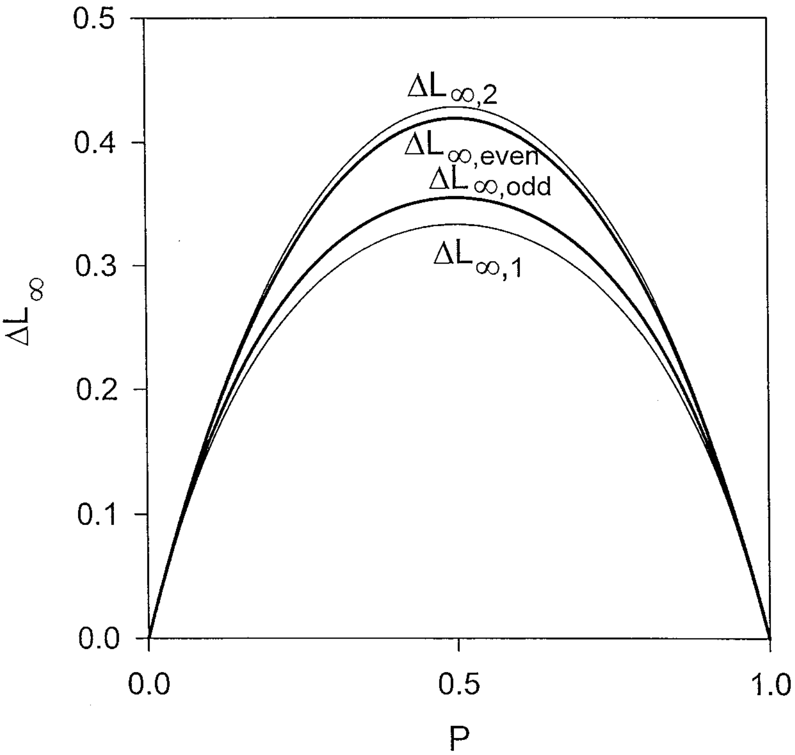

Fig. 4. Fractal amplitudes for a periodic process in which only the spatial rescaling factor alternates. We assume the probability for digging holes Q to be zero; hence, we can consider λ alternating between $\lambda_1 = 2$ and $\lambda_2 = 4$. The plot is symmetric and fractal amplitude attraction takes place for any probability P of putting a hill on the surface.

Namely, we have assumed that the deposition parameters are time-dependent with period 2. We let the probabilities for depositing hills, P, and holes, Q, alternate between two sets of values, while keeping the rescaling factor λ fixed, and another case is that of alternating λ and constant P and Q. In all cases we have shown that the main property of the random deposition model—its logarithmic fractal surface roughness—remains invariant under periodic perturbation.

We have compared our results with those for the static case and interpreted them in terms of an interaction. In the general case $P \neq 0 \neq Q$ periodically varying parameters lead to amplitude interactions which may be repulsive, attractive, or neutral, depending on the specific choice of parameters.

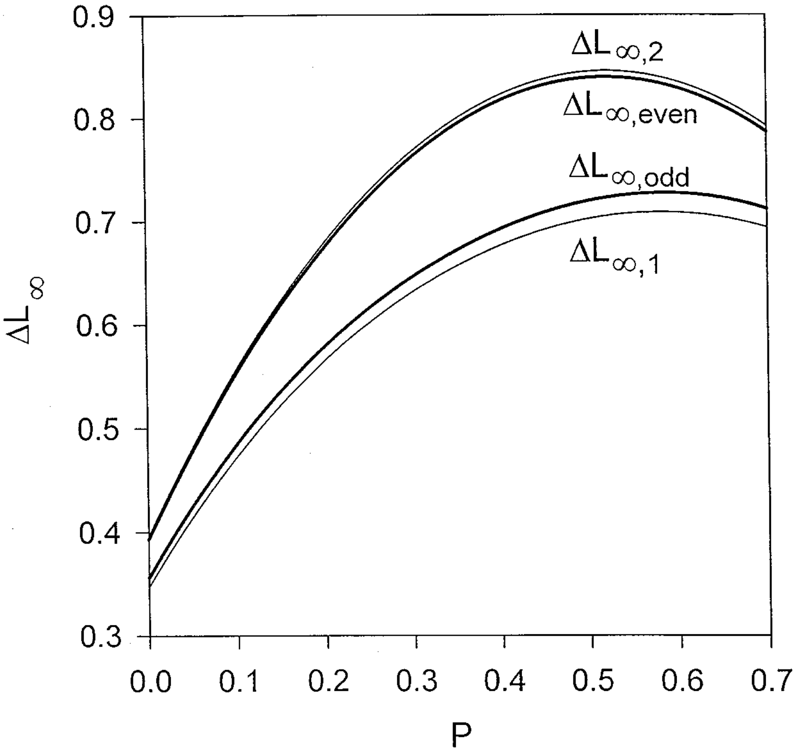

Fig. 5. Fractal amplitudes for a periodic process with λ alternating between $\lambda_1 = 3$ and $\lambda_2 = 8$ and nonzero probability of digging a hole $Q = 0.3$. The physical range for probability P of depositing a hill is $0 \leqslant P \leqslant 0.7$. The fractal amplitudes attract for all P.

We have extended the exact results previously obtained [6] by calculating the asymptotic amplitudes for arbitrary alternating rescaling factors, λ_1 and λ_2, with $\lambda_1 \geqslant 3$ and $\lambda_2 \geqslant 3$. We have shown that alternating rescalings lead to amplitude attraction, if P and Q are assumed to be constant.

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